Appendix A

Notation

- 3×1 vectors are represented in **bold**. Example: $\boldsymbol{\omega}$
- The scalar component of a vector is not in bold and has a subscript. Example: ω_{x}
- 3×3 matrices are shown in [brackets]. Example: $[R]$
- The absolute value of a scalar is shown using one vertical bar on either side of the |variable|. Example: $|B_{\mathrm{x}}|$
- The magnitude of a vector is shown with two vertical bars on either side of the ||**variable**||. Example: $||\mathbf{B}||$
- The 3×3 identity matrix is represented by $[I_{3\times3}]$.
- The transpose of a matrix is represented by a superscript T. Example: $[R]^T$
- The trace matrix operation is represented by tr(). Example: $\mathrm{tr}([R])$
- The reference frame of a vector is represented by superscript calligraphy letter before the vector. Example: ${}^{\mathcal{I}}\mathbf{r}$
- The skew-symmetric matrix operator is represented by brackets around the variable and a cross product within the brackets. Example: $[\boldsymbol{\omega}\times]$
- The inertial time derivative $\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}$ of a variable is represented by a dot above the variable. Example: $\frac{{}^{\mathcal{I}}\mathrm{d}}{\mathrm{d}t}\boldsymbol{\omega} = \dot{\boldsymbol{\omega}}$

Appendix B

Explicit Runge-Kutta Integrator Definitions

The family of explicit Runge-Kutta numeric integrators is generalized as follows (repeated from Section 8.1.7.1):

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i \, k_i \tag{8.26}$$

where

$$\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + c_2 h, y_n + a_{21} k_1) \\
k_3 &= f(t_n + c_3 h, y_n + a_{31} k_1 + a_{32} k_2) \\
&\vdots \\
k_s &= f(t_n + c_s h, y_n + a_{s1} k_1 + a_{s2} k_2 + \ldots + a_{s,s-1} k_s).
\end{aligned}$$

A specific Runge-Kutta integrator are is given by its Butcher tableau, which is a standard form of presenting the coefficients used by Equation 8.26. The general form of a Butcher tableau for an explicit Runge-Kutta integrator is shown in Table B.1. The Butcher tableau of each Runge-Kutta integrator used within this dissertation is shown in Tables B.2 through B.7.

Table B.1: The general form of the Butcher tableau for explicit Runge-Kutta Methods [34].

0				
c_2	a_{21}			
c_3	a_{31}	a_{32}		
$\vdots$	$\vdots$	$\vdots$	$\ddots$	
c_s	a_{s1}	a_{s2}	$\dots$	$a_{s,s-1}$
	b_1	b_2	$\dots$	b_{s-1}

Table B.2: The Butcher tableau for explicit fixed RK2 (midpoint method) [34].

0		
1/2	1/2	
	0	1

Table B.3: The Butcher tableau for explicit fixed RK3 (Kutta method) [21].

0			
1/2	1/2		
1	-1	2	
	1/6	2/3	1/6

Table B.4: The Butcher tableau for explicit fixed RK4 (Runge-Kutta method) [34].

0				
1/2	1/2			
1/2	0	1/2		
1	0	0	1	
	1/6	2/6	2/6	1/6

Table B.5: The Butcher tableau for explicit fixed RK5 (fixed Dormand-Prince method) [34].

0						
1/5	1/5					
3/10	3/40	9/40				
4/5	44/45	-56/15	32/9			
8/9	19372/6561	-25360/2187	64448/6561	-212/729		
1	9017/3168	-355/33	46732/5247	49/176	-5103/18656	
	35/384	0	500/1113	125/192	-2187/6784	11/84

Table B.6: The Butcher tableau for explicit fixed RK6 (Hammund scheme) [3].

0							
4/7	4/7						
5/7	115/112	-5/16					
6/7	589/630	5/18	-16/45				
(5-$\sqrt{5}$)/10	229/1200-29$\sqrt{5}$/6000	119/240-187$\sqrt{5}$/1200	-14/75+34$\sqrt{5}$/375	-3$\sqrt{5}$/100			
(5+$\sqrt{5}$)/10	71/2400-587$\sqrt{5}$/12000	187/480-391$\sqrt{5}$/2400	-38/75+26$\sqrt{5}$/375	27/80-3$\sqrt{5}$/400	1+$\sqrt{5}$/4		
1	-49/480+43$\sqrt{5}$/160	-425/96+51$\sqrt{5}$/32	52/15+4$\sqrt{5}$/5	-27/16+3$\sqrt{5}$/16	5/4-3$\sqrt{5}$/4	5/2-$\sqrt{5}$/2	
	1/12	0	0	0	5/12	5/12	1/12

Table B.7: The Butcher tableau for explicit fixed RK7 (fixed Fehlburg method) [34].

0												
2/27	2/27											
1/9	1/36	1/12										
1/6	1/24	0	1/8									
5/12	5/12	0	-25/16	25/16								
1/2	1/20	0	0	1/4	1/5							
5/6	-25/108	0	0	125/108	-65/27	125/54						
1/6	31/300	0	0	0	61/225	-2/9	13/900					
2/3	2	0	0	-53/6	704/45	-107/9	67/90	3				
1/3	-91/108	0	0	23/108	-976/135	311/54	-19/60	17/6	-1/12			
1	2383/4100	0	0	-341/164	4496/1025	-301/82	2133/4100	45/82	45/164	18/41		
	41/840	0	0	0	0	34/105	9/35	9/35	9/280	9/280	41/840	